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Asymptotically Normal Dynamical Semigroups

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A definition and a characterization of asymptotically normal dynamical systems are given. In particular, a theorem concerning the return to equilibrium is presented.

KEY WORDS: Dynamical systems; detailed balance condition; asymptotically normal dynamics; return to equilibrium.

1. INTRODUCTION

The theory of quantum dynamical semigroups provides a convenient mathematical description of the irreversible dynamics of an open quantum system. Among the problems in this theory that have receive much attention are the following: conditions for a dynamical semigroup to induce approach to a stationary (equilibrium) state and the study of related subjects, such as ergodic theorems. In this paper I present general theorems of ergodic type and on approach the equilibrium for asymptotically normal dynamical semigroups.

2. PRELIMINARIES AND ASSUMPTIONS

Let $(\mathscr{M}, \tau_t, \omega)$ be a dynamical system, i.e., $\mathscr{M}$ is a W^*-algebra on a Hilbert space $\mathscr{H}$, τ_t is a strongly positive dynamical semigroup on $\mathscr{M}$, and ω is a τ_t-invariant state on $\mathscr{M}$ such that $\omega(A) = (\Omega, A\Omega)$ for $A \in \mathscr{M}$, and $\Omega \in \mathscr{H}$ is a cyclic and separating vector for $\mathscr{M}$.

I restrict myself to the class of dynamical systems satisfying the following two conditions:

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Condition I (Detailed balance condition[1]). A normal faithful state ω of $\mathscr{M}$ satisfies the detailed balance condition with respect to a dynamical semigroup τ_t whenever

$$\omega \circ \tau_t = \omega, \qquad t \geqslant 0 \tag{1}$$

$$\omega(A^* \tau_t(B)) = \omega(\sigma(B^*)\, \tau_t \sigma(A)), \qquad A, B \in \mathscr{M}, \quad t \geqslant 0 \tag{2}$$

where σ denotes a reversing operation of $\mathscr{M}$, i.e., $\sigma\colon \mathscr{M} \to \mathscr{M}$ is an antilinear Jordan automorphism of order two. This condition of detailed balance originated from the quantum theory of atoms and molecules and says that every pair of energy levels, with the probabilities of transitions between them, constitutes a balanced system.

Condition II. A dynamical system $(\mathscr{M}, \tau_t, \omega)$ is called asymptotically normal if

$$\lim_{t \to +\infty} \omega(\tau_t(A)\, \tau_t(B)) = \lim_{t \to +\infty} \omega(\tau_t^\sigma(A)\, \tau_t^\sigma(B)) \tag{3}$$

for $A, B \in \mathscr{M}$, where $\tau_t^\sigma = \sigma \circ \tau_t \circ \sigma$ and σ is the reversing operation.

The idea behind this condition is simple. τ_t represents the dynamics of an open system and the Markovian approximation is assumed. Hence, it seems natural to consider τ_t as a reduction of the Hamiltonian evolution. Further, one can expect that τ_t should inherit some features of the Hamiltonian dynamics. In particular, let us observe that condition II is fulfilled for the Hamiltonian evolution. So, it is assumed that the dynamical system $(\mathscr{M}, \tau_t, \omega)$ inherits from the Hamiltonian evolution the property of condition II.

3. RESULTS

(i) Let $(\mathscr{M}, \tau_t, \omega)$ be a dynamical system such that conditions I and II are satisfied. Then the induced semigroup $\hat{\tau}_t$ $[\hat{\tau}_t A\Omega \stackrel{\text{df}}{=} \tau_t(A)\,\Omega]$ has the following property:

$$\lim_{t \to +\infty} (A\Omega, \hat{\tau}_t^* \hat{\tau}_t B\Omega) = \lim_{t \to +\infty} (A\Omega, \hat{\tau}_t \hat{\tau}_t^* B\Omega) \tag{4}$$

Let us observe that (4) gives the justification for the name of the considered dynamical systems. Further, if $\hat{\tau}_t$ is a semigroup of normal operators on $\mathscr{H}$, then (4) holds. However, there are nontrivial semigroups for which (4) is true. The simplest example is provided by the semigroup $T_t = e^{(iH - D)t}$ on a finite-dimensional Hilbert space $\mathscr{H}$, where H and D are self-adjoint operators on $\mathscr{H}$ and $(Df, f) > 0$ for each $f \in \mathscr{H}$.

(ii) Let $(\mathscr{M}, \tau_t, \omega)$ be a dynamical system such that conditions I and II are satisfied. Then

$$\lim_{t \to +\infty} \hat{\tau}_t^* \hat{\tau}_t = P \tag{5}$$

where P is an orthogonal projector.

Sketch of Proof. First, let us remark that for the contraction semigroup $\hat{\tau}_t$

$$s\text{-lim}\, \hat{\tau}_t^* \hat{\tau}_t = P \tag{6}$$

always exists (cf. proof in Ref. 2, p. 40) and obviously P is a positive operator on $\mathscr{H}$.

Second, condition II implies

$$s\text{-}\lim_{t \to +\infty} \hat{\tau}_t^* \hat{\tau}_t = s\text{-}\lim_{t \to +\infty} \hat{\tau}_t \hat{\tau}_t^* = P \tag{7}$$

Moreover,

$$\hat{\tau}_s P \hat{\tau}_s^* = P = \hat{\tau}_s^* P \hat{\tau}_s \tag{8}$$

for any $s > 0$. Therefore, one can prove

$$P = P^3 \tag{9}$$

Hence, it is easy to see that P is an orthogonal projector.

(iii) In order to formulate the next result, let us denote the orthogonal projector on the subspace of $\mathscr{H}$ formed by the vectors invariant under all $\hat{\tau}_t$ by E, and the mean value by M_τ; let

$$\mathscr{N}(\tau) = \{A \in \mathscr{M}; \tau_t(A^*A) = \tau_t(A)^* \tau_t(A), \tau_t(AA^*) = \tau_t(A)\, \tau_t(A)^*; t \geqslant 0\}$$

Let the orthogonal projector on the subspace $[\mathscr{N}(\tau)\, \Omega]$ be F, and let $l_\psi(A) = (\psi, A\Omega)$ for $\psi \in \mathscr{H}$, $A \in \mathscr{M}$.

I now present a theorem of ergodic type.

Theorem 1. Consider the following conditions:

$$\lim_{t \to +\infty} (\psi, (\hat{\tau}_t \hat{\tau}_t^* - E)\, \psi) = 0 \tag{10}$$

$$\lim_{t \to +\infty} \|l_\psi \circ \tau_t - l_\psi \circ M_\tau\| = 0 \tag{11}$$

$$(E - F)\, \psi = 0 \tag{12}$$

It follows that $(10) \Rightarrow (11) \Rightarrow (12)$.

Conversely, if conditions I and II are satisfied [for a dynamical system $(\mathcal{M}, \tau_t, \omega)$], then (12) ⇒ (10).

Remarks. Implications (10) ⇒ (11) ⇒ (12) were proven by Robinson.[3] Equivalences (11) and (12) under stronger assumptions are a local version of Frigerio's result.[4]

Sketch of Proof. In order to prove (12) ⇒ (10), it is enough to modify the Robinson argument and use (ii).

(iv) The next result ascertains that an asymptotic normal dynamical system $(\mathcal{M}, \tau_t, \omega)$ has the fundamental property of return to equilibrium. This is to say that for any normal state φ the time evolution of φ_t has a limit as t tends to infinity.

Theorem 2. Let $(\mathcal{M}, \tau_t, \omega)$ be a dynamical system such that conditions I and II are satisfied. Moreover, let

$$\lim_{t \to +\infty} \omega(A\tau_t(B)) \tag{13}$$

exist for $A \in \mathcal{M}$ and $B \in \mathcal{N}(\tau)$. Then the system $(\mathcal{M}, \tau_t, \omega)$ manifests a return to equilibrium, i.e., the limit

$$\varphi_+(A) = \lim_{t \to +\infty} \varphi(\tau_t(A)) \tag{14}$$

exists for all $A \in \mathcal{M}$ and all normal states on $\mathcal{M}$.

The result (ii) is the crucial point in the proof of Theorem 2.

(v) Now, we consider the question of the connection between conditions I and II. The answer is rather surprising. Namely, let a dynamical system $(\mathcal{M}, \tau_t, \omega)$ satisfy the detailed balance condition. Then condition II is equivalent to the following property of the reversing operation:

$$\sigma \circ q \circ \sigma = q \tag{15}$$

where q is the positive map on $\mathcal{M}$ defined by the equality

$$q(A)\,\Omega = PA\Omega \tag{16}$$

where $P = s\text{-}\lim_{t \to +\infty} \hat{\tau}_t^* \hat{\tau}_t$ [cf. the first remark in the proof of (ii)]. In other words, condition II picks out the special reversing operation in the family of all admissible reversing operations. Therefore, one can say that conditions I and II are complementary to one another.

Finally, sophisticated algebraic results give a possibility of constructing examples of nontrivial models of asymptotically normal dynamical systems.

Although these examples are only of a rather mathematical nature, they indicate that dynamical systems with conditions I and II may well exist in nature. These examples and technical details underlying this paper will be presented elsewhere.

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